FINITE DIFFERENCE APPROXIMATION OF THE DEFLECTION EQUATIONS

OF A CONICAL SHELL WITH CLOSED BASE

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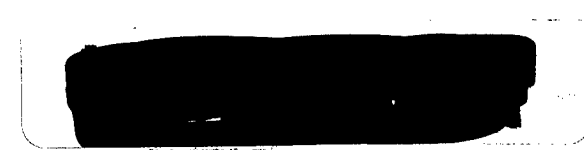
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I. INTRODUCTION

In this report we give finite difference approximations to the differential equations for the following problems: (1) the bending of a variable-thickness truncated right conical shell attached to an axially and sectionally symmetric base plate of variable thickness, (2) the bending of two joined variable-thickness truncated right conical shells.

In case (1) above, the terms arising from consideration of the transverse deflection of the base caused by radial loading have been included. As a consequence the finite difference equations are non-linear. For solutions of these equations we suggest an iterative scheme combined with a direct method based on the factorization theorem for square matrices. In case (2), the finite difference equations are linear and procedures for solution are straightforward.

II. DIFFERENTIAL EQUATIONS

The differential equations for the conical shells are derived in TR 1-21 [1]* and are repeated below for convenience. Coordinate system, geometry, and variables are shown in Fig. 1. The differential equations are given in terms of the variables S and θ where S is related to the force per unit length N_φ (see Fig. 2) by:

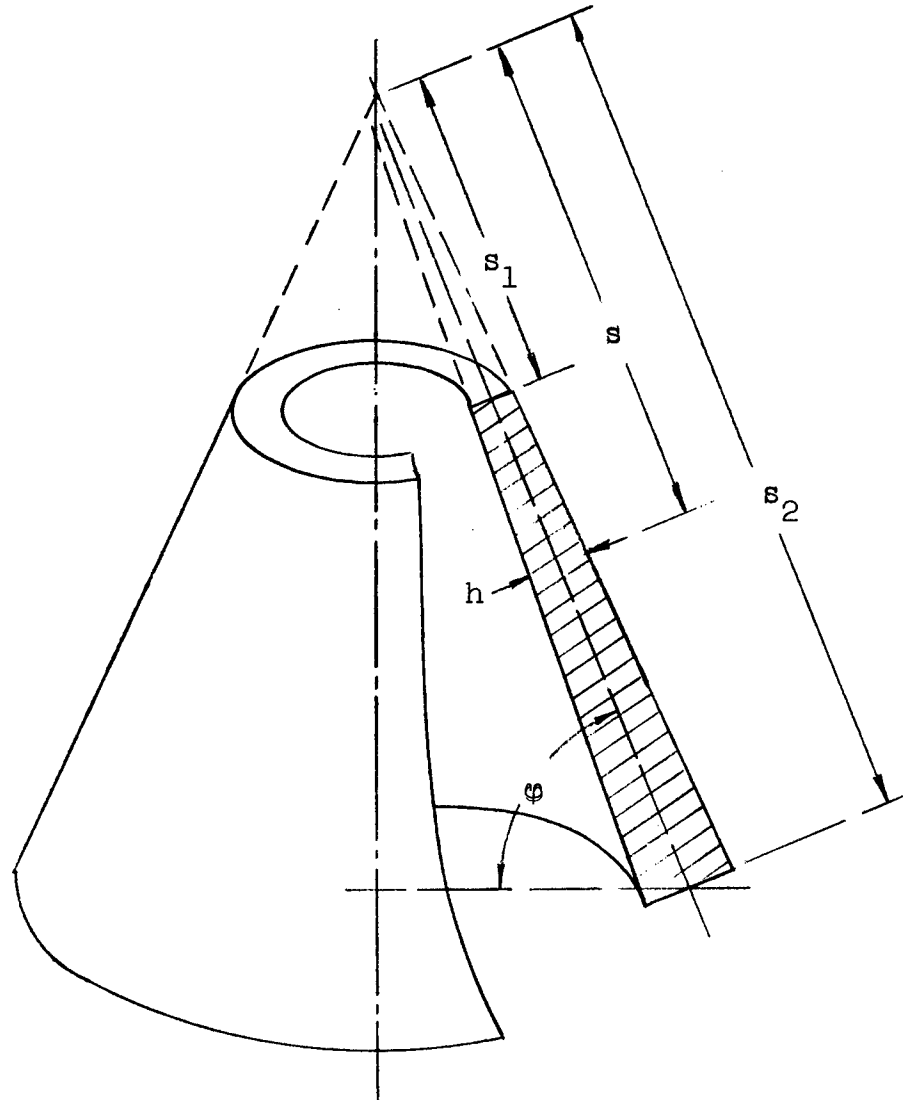* Numbers in square brackets refer to bibliography on Page 29.

Fig. 1
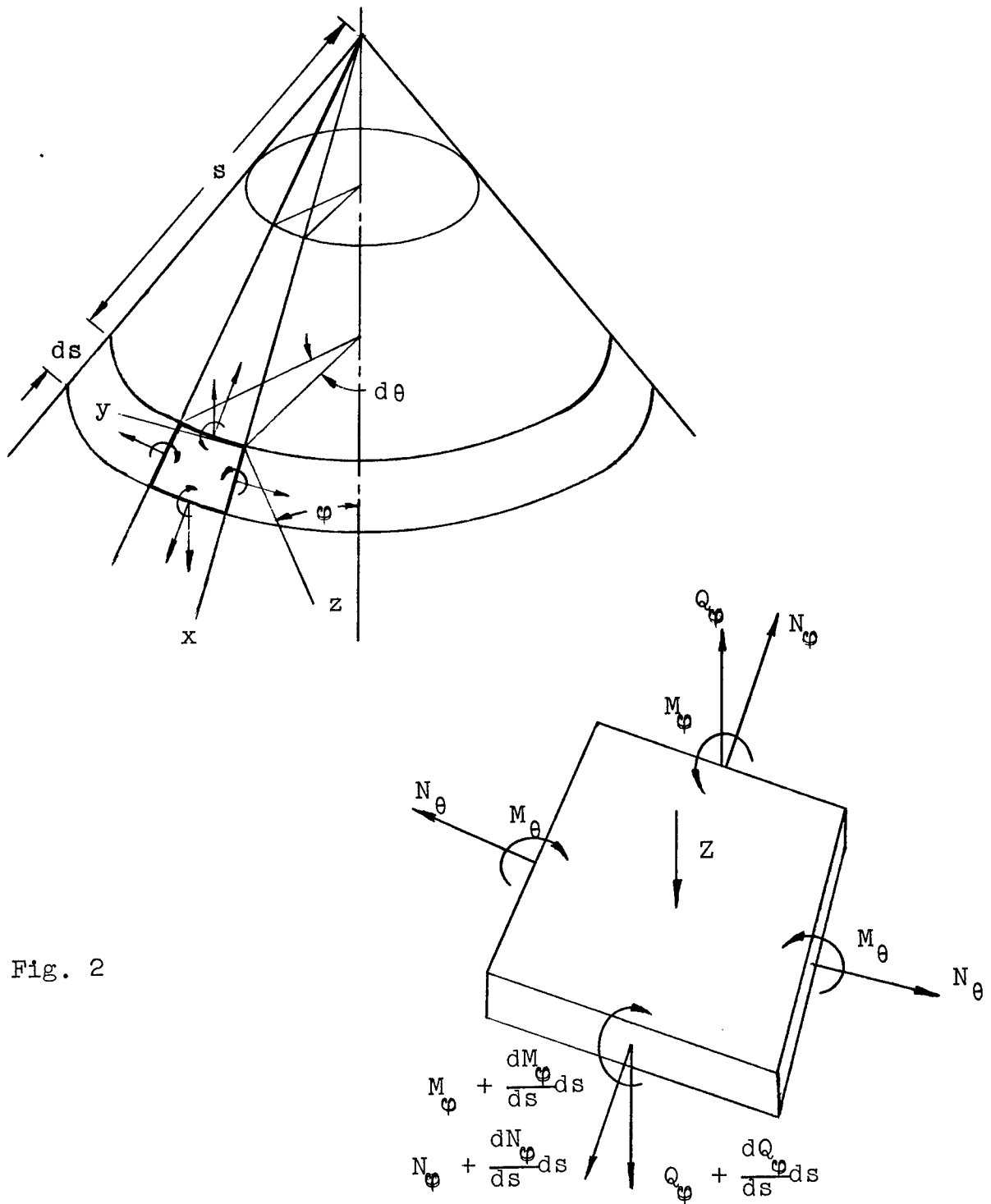
Fig. 2

2.1 $\qquad S = \dfrac{sN}{h^2}$

and θ is the slope of the median plane in the axial direction, i.e.:

2.2 $\qquad \theta = \dfrac{dw}{ds}$

The differential equations are ([1] p. 16):

2.3 $\qquad L(\theta) + f_1\theta = \lambda_1 S + F(s)$

2.4 $\qquad L(S) + f_2 S = \lambda_2\theta$

where the differential operator L is of second order:

2.5 $\qquad L(U) = h\,\cot\varphi\left[sU'' + (1 + 3s\,\dfrac{h'}{h})U' - \dfrac{U}{s}\right]$

The apostrophe represents differentiation with respect to s. The other quantities appearing in the differential equations are defined as follows:

2.6 $\qquad f_1 = s\nu h'\cot\varphi$

where ν is Poisson's ratio,

2.7 $\qquad f_2 = \left[(2 + \nu)h' + 2sh''\right]\cot\varphi$

2.8 $\qquad \lambda_1 = \dfrac{12(1 - \nu^2)}{E}$

where E is the modulus of elasticity,

2.9 $\qquad \lambda_2 = -E$

and

2.10 $\qquad F(s) = \dfrac{12(1 - \nu^2)}{E}\ \dfrac{F(s)}{h^2\cot\varphi}$

where ([1] p. 4):

$$F(s) = -\left[sN_\varphi \sin\varphi + s\,Q_\varphi \cos\varphi\right] = \int_{s_0}^{s} Z\cos\varphi\ sds$$

2.11

$$-\left[sN_\varphi \sin\varphi + sQ_\varphi \cos\varphi\right]_{s=s_0}$$

and Z is a distributed load normal to the middle surface of
the shell and acting over the whole surface, and Q_φ is a shear
force per unit length acting normal to the middle surface of
the shell (see Fig.2)

The differential equations for the base may be written
as follows [2]:

2.12 $\qquad (D\phi')' + \frac{D}{r}\phi' + (\frac{v}{r}D - \frac{D}{r^2} - \sigma_r(a)t)\phi = Q(r)$

2.13 $\qquad r^2\psi'' + (1 - \frac{r}{t}t')r\psi' + (\frac{vr}{t}t' - 1)\psi = 0$

where the apostrophe represents differentiation with respect
to r (the radial distance from the center of the plate),
$t(r)$ is the thickness of the plate, $D = E(t(r))^3/12(1 - v^2)$,
$v(r)$ is the transverse deflection of the plate, $\sigma_r(r)$ is the
membrane stress in the plate, q is a uniformly distributed
load acting transversely upon the plate, and

2.14 $\qquad Q(r) = \frac{qr}{2}$

2.15 $\qquad \frac{\psi(a)}{r} = \sigma_r(a)t(r)$

and

2.16 $\qquad \phi = \frac{dv}{dr}$

Coordinate system, geometry, and variables are shown in Fig. 3.

For the case of the bending of two joined cones the dif-
ferential equation for each cone is given by corresponding sets
of equations 2.3 and 2.4 along with definitions 2.5, 2.6,
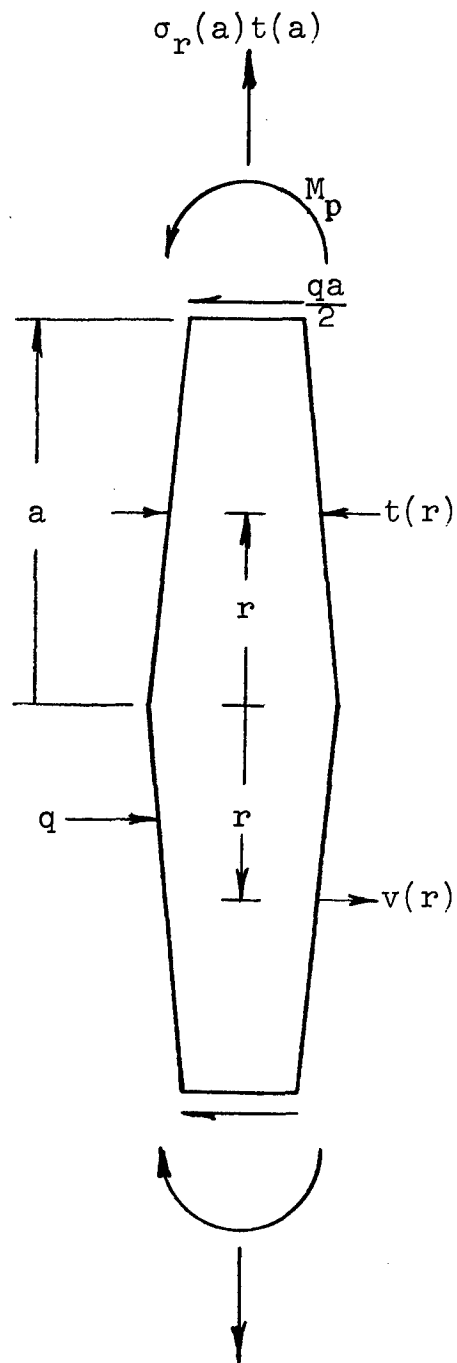2.7, 2.8, 2.9, 2.10 and 2.11.

Fig. 3

DIAMETRAL SECTION OF BASE-PLATE

III. BOUNDARY AND JUNCTION CONDITIONS

For the cone with attached base-plate the boundary conditions at the free end of the cone are

3.1 $\quad\quad M_\varphi = M$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ at open end of cone

3.2 $\quad\quad \epsilon_{ym} = \epsilon$

and at the center of the base-plate are

3.3 $\quad\quad \Phi = 0$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ at center of base-plate

3.4 $\quad\quad \psi = 1$

where M_φ is moment per unit length and ϵ_{ym} is the strain of the middle surface of the cone in the y direction (circumferentially).

In terms of the variables of 2.3 and 2.4 the boundary conditions 3.1 and 3.2 may be written

3.5 $\quad\quad \dfrac{Eh^3}{12(1 - \nu^2)} (\theta' + \dfrac{\nu}{s}\theta) = M$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ at open end of cone

3.6 $\quad\quad h(2h' - \dfrac{\nu h}{s})S + h^2 S' = \epsilon$

At the junction of the cone and the base plate the following conditions may be prescribed

3.7 $\qquad \sigma_r t = V$

3.8 $\qquad M_p = M_{p}$

3.9 $\qquad -\Phi = \theta$

3.10 $\qquad \epsilon_p = \epsilon_{ym}$

at the junction of the cone and base-plate

where M_p and ϵ_p are respectively the moment per unit length and the strain along the circumference of the base-plate.

In terms of the variables of 2.3, 2.4, 2.12 and 2.13 the junction conditions can be written

3.11 $\qquad \dfrac{\psi}{r} = P \tan\eta - \dfrac{h^2 S}{s \cos\eta}$

3.12 $\qquad \dfrac{Et^3}{12(1 - \nu^2)}(\Phi' + \dfrac{\nu}{r}\Phi) = \dfrac{Eh^3}{12(1 - \nu^2)}(\theta' + \dfrac{\nu}{s}\theta)$

3.13 $\qquad -\Phi = \theta$

3.14 $\qquad \dfrac{1}{Et}(\psi' - \dfrac{\nu}{r}\psi) = h(2h' - \dfrac{\nu h}{s})S + h^2 S'$

at junction of cone and base-plate

where $P = Q = \dfrac{qa}{2}$

For the case of the joined cones the conditions at the free ends are

3.15 $\qquad M_{\varphi 1} = M_1$

3.16 $\qquad M_{\varphi 2} = M_2$

3.17 $\qquad \epsilon_{ym1} = \epsilon_1$

3.18 $\qquad \epsilon_{ym2} = \epsilon_2$

and at the joined ends are ([1], p. 20)

3.19
$$\left[(2hh' - \nu\frac{h^2}{s})S + h^2 S'\right]_1 = \left[(2hh' - \nu\frac{h^2}{s})S + h^2 S'\right]_2$$

3.20
$$\theta_1 = \theta_2$$

3.21
$$\left[\frac{Eh^3}{12(1 - r^2)}(\theta' + \nu\frac{\theta}{s})\right]_1 = \left[\frac{Eh^3}{12(1 - \nu^2)}(\theta' + \nu\frac{\theta}{s})\right]_2$$

3.22
$$\left(P\tan\varphi - \frac{h^2 S}{s\cos\varphi}\right)_1 = \left(P\tan\varphi - \frac{h^2 S}{s\cos\varphi}\right)_2$$

where the subscripts 1 and 2 refer to respective joined cones.

IV. FINITE DIFFERENCE SCHEME

We will approximate the differential equations for the cone at equally spaced net points s_0, s_1, s_2, ----, s_I where

$$s_{i+1} - s_i = \Delta s$$

We will approximate the differential equations for the base plate by equally spaced net points r_0, r_1, r_2, ----, r_J where

$$r_{j+1} - r_j = \Delta r$$

We will append a subscript to each variable to designate its corresponding approximation at a point. For example, $S(s_i) = S_i$, $\Phi(r_j) = \Phi_j$, and similarly for the remaining variables. Derivatives will be obtained by a Taylor's series expansion about surrounding net points, neglecting terms $O((\Delta s)^2)$ and $O((\Delta r)^2)$.

The difference approximations for the cone equations are

4.1 $\qquad L(\theta_i) + F_{1i}\, \theta_i = \lambda_1\, S_i + F_i$

4.2 $\qquad L(S_i) + F_{2i}\, S_i = \lambda_2\, \theta_i$

where

4.3 $\qquad F_{1i} = s_i \nu \dfrac{h_{i+1} - h_{i-1}}{2\Delta s} \cot\alpha$

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$$4.4 \qquad F_{21} = \left\lceil (2 + \nu) \frac{h_{i+1} - h_{i-1}}{2\Delta s} \right.$$

$$+ (2s_i) \frac{h_{i-1} - 2h_i + h_{i+1}}{(\Delta s)^2} \left. \right\rceil \cot\varphi$$

$$4.5 \qquad F_1 = \frac{12(1 - \nu^2)}{E} \sum_{s=s_o}^{s_1} Z \cos\varphi \, s \Delta s - s_o \left\lceil \frac{(N_\varphi)_o \sin\varphi}{h_o^2 \cot\varphi} \right.$$

$$+ \frac{(Q_\varphi)_o \cos\varphi}{h_o^2 \cot\varphi} \left. \right\rceil$$

and the difference approximation for the differential operator is

$$4.6 \qquad L(U_i) = h_i \cot\varphi \left\lceil s_i U_i'' + (1 + 3s_i \frac{h_i'}{h_i})U_i' - \frac{U_i}{s_i} \right\rceil$$

The first and second derivatives are approximated as usual by

$$4.7 \qquad V_i' = \frac{V_{i+1} - V_{i-1}}{2\Delta s} + O((\Delta s)^2)$$

and

$$4.8 \qquad V_i'' = \frac{V_{i-1} - 2V_i + V_{i+1}}{(\Delta s)^2} + O((\Delta s)^2)$$

Substituting 4.7 and 4.8 in 4.6 gives

$$L(U_i) = h_i \cot\varphi \left\lceil \left\{ \frac{s_i}{(\Delta s)^2} - (1 + \frac{3s_i(h_{i+1} - h_{i-1})}{2\Delta s}) \frac{1}{2\Delta s} \right\} U_{i-1} \right.$$

4.9
$$- \left\{ \frac{2s_1}{(\Delta s)^2} + \frac{1}{s_1} \right\} U_1$$

$$+ \left\{ \frac{s_1}{(\Delta s)^2} + \left(1 + \frac{3s_1(h_{1+1} - h_{1-1})}{2\Delta s} \right) \frac{1}{2\Delta s} \right\} U_{1+1} \bigg]$$

The difference approximations for the base-plate equations are

4.10
$$(D_j \phi_j')' + \frac{D_j}{r_j} \phi_j' + \left(\frac{\nu}{r_j} D_j' - \frac{D_j}{r_j^2} - (\sigma_r)_j t_j \right) \phi_j = Q_j$$

4.11
$$r_j^2 \psi_j'' + \left(1 - \frac{r_j}{t_j} t_j' \right) r_j \psi_j' + \left(\frac{\nu r_j}{t_j} t_j' - 1 \right) \psi_j = 0$$

where

4.12
$$D_j = \frac{E t_j^3}{12(1 - \nu^2)}$$

Substituting 4.7 and 4.8 in 4.10 and 4.11 gives

$$\left\{ \frac{D_{j+1} - D_{j-1}}{(2\Delta r)^2} + \frac{D_j}{(\Delta r)^2} + \frac{D_j}{2r_j \Delta r} \right\} \phi_{j+1}$$

4.13
$$+ \left\{ \frac{\nu}{r_j} \frac{D_{j+1} - D_{j-1}}{2\Delta r} - \frac{D_j}{r_j^2} - \sigma_r(a) t_j - \frac{2D_j}{(\Delta r)^2} \right\} \phi_j$$

$$+ \left\{ \frac{D_{j-1} - D_{j+1}}{(2\Delta r)^2} + \frac{D_j}{(\Delta r)^2} - \frac{D_j}{2r_j \Delta r} \right\} \phi_{j-1} = Q_j$$

and

$$\left\{ \frac{r_j^2}{(\Delta r)^2} + \left(1 - \frac{r_j}{t_j} \left[\frac{t_{j+1} - t_{j-1}}{2\Delta r} \right] \right) \frac{r_j}{2\Delta r} \right\} \psi_{j+1}$$

4.14 $$+ \left\{ \frac{\nu \, r_j}{t_j} \left[\frac{t_{j+1} - t_{j-1}}{2\Delta r} \right] - 1 - \frac{2 \, r_j^2}{(\Delta r)^2} \right\} \psi_j$$

$$+ \left\{ \frac{r_j^2}{(\Delta r)^2} - \left(1 - \frac{r_j}{t_j} \left[\frac{t_{j+1} - t_{j-1}}{2\Delta r} \right] \right) \frac{r_j}{2\Delta r} \right\} \psi_{j-1} = 0$$

For the case of the cone with attached base-plate we first investigate the difference approximations of the differential equations at their boundaries and junction, i.e., where $i = 0$ and I, and where $j = 0$ and J.

For the cone when $i = 0$ the variables θ_{-1}, θ_0, θ_1, and S_0 occur in equation 4.1 and the variables S_{-1}, S_0, S_1, and θ_0 occur in equation 4.2. These two relations, i.e., 4.1 and 4.2 when $i = 0$, will be denoted

4.15 $\qquad H_1(\theta_{-1}, \theta_0, \theta_1, S_0) = k_1$

and

4.16 $\qquad H_2(S_{-1}, S_0, S_1, \theta_0) = k_2$

For the base-plate when $j = J$ the variables ϕ_{J-1}, ϕ_J,

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ϕ_{J+1} occur in equation 4.10 and the variables ψ_{J-1}, ψ_J, ψ_{J+1} occur in equation 4.11. These two relations, i.e., 4.10 and 4.11 when $j = J$, will be denoted

4.17 $\qquad H_3(\phi_{J-1}, \phi_J, \phi_{J+1}) = k_3$

and

4.18 $\qquad H_4(\psi_{J-1}, \psi_J, \psi_{J+1}) = k_4$

In 4.15 and 4.16 the variables θ_{-1} and S_{-1} are located outside the junction of the cone and base-plate; similarly in 4.17 and 4.18 the variables ϕ_{J+1} and ψ_{J+1} are located outside the junction of the cone and base-plate. These four variables can be eliminated by means of the junction conditions 3.12 and 3.14 which in difference form for $i = 0$, $j = J$ are

4.19 $\qquad \dfrac{E t_j^3}{12(1 - \nu^2)} \left(\dfrac{\phi_{J+1} - \phi_{J-1}}{2\Delta r} + \dfrac{\nu}{a} \phi_J \right)$

$$= \dfrac{E h_o^3}{12(1 - \nu^2)} \left(\dfrac{\theta_1 - \theta_{-1}}{2\Delta s} + \dfrac{\nu}{s_o} \theta_o \right)$$

4.20 $\quad \dfrac{1}{E t_j}\left(\dfrac{\psi_{J+1} - \psi_{J-1}}{2\Delta r} - \dfrac{\nu}{a} \psi_J \right) = h_o S_o \left(2\dfrac{h_1 - h_{-1}}{2\Delta s} - \dfrac{\nu h_o}{s_o} \right) + h_o^2 \dfrac{S_1 - S_{-1}}{2\Delta s}$

Solving 4.15 and 4.16 for θ_{-1} and S_{-1} and substituting in the right hand side of 4.19 and 4.20 respectively and solving 4.17 and 4.18 for ϕ_{J+1} and ψ_{J+1} and substituting in the left-hand side of 4.19 and 4.20 respectively gives two equations in the

variables θ_0, θ_1, Φ_J, Φ_{J-1} and S_0, S_1, ψ_J, ψ_{J-1}. These equations will be denoted

4.21 $\qquad H_5(\theta_0,\ \theta_1, \Phi_J,\ \Phi_{J-1}) = k_5$

4.22 $\qquad H_6(S_0,\ S_1,\ \psi_J,\ \psi_{J-1}) = k_6$

We next eliminate θ_0 and S_0 from 4.21 and 4.22 by means of the remaining junction conditions 3.11 and 3.13. In finite difference form for $i = 0$, $j = J$ these junction conditions become

4.23 $\qquad \dfrac{\psi_J}{a} = P\ \tan\varphi - \dfrac{h_0^2\ S_0}{S_0\ \cos\varphi}$

4.24 $\qquad -\ \Phi_J = \theta_0$

Solving 4.23 and 4.24 for S_0 and θ_0 respectively and substituting in 4.22 and 4.21 respectively gives two equations in the variables θ_1, Φ_{J-1}, Φ_J and S_1, ψ_{J-1}, ψ_J. These equations may be denoted

4.25 $\qquad H_7(\Phi_J,\ \Phi_{J-1},\ \theta_1) = k_7$

4.26 $\qquad H_8(\psi_J,\ \psi_{J-1},\ S_1) = k_8$

Now we turn to the boundary conditions. Setting $i = I$ in equations 4.1 and 4.2 gives two equations which we denote as

4.27 $H_9(\theta_{I-1}, \theta_I, \theta_{I+1}, S_I) = k_9$

4.28 $H_{10}(S_{I-1}, S_I, S_{I+1}, \theta_I) = k_{10}$

The outside points θ_{I+1} and S_{I+1} can be eliminated by means of the boundary conditions 3.5 and 3.6. In difference form when $i = I$ these conditions are

4.29 $\dfrac{E\, h_I^3}{12(\,1 - \nu^2)} \left(\dfrac{\theta_{I+1} - \theta_{I-1}}{2\Delta s} + \dfrac{\nu}{S_I}\, \theta_I\right) = M$

4.30 $h_I\left(2\dfrac{h_{I+1} - h_{I-1}}{2\Delta s} - \dfrac{\nu h_I}{s_I}\right)S_I + h_I^2\left(\dfrac{S_{I+1} - S_{I-1}}{2\Delta s}\right) = \epsilon$

Solving 4.29 for θ_{I+1} and 4.31 for S_{I+1} and substituting in 4.27 and 4.28 respectively gives two equations which we will denote as

4.31 $H_{11}(\theta_I, \theta_{I-1}, S_I) = k_{11}$

4.32 $H_{12}(S_I, S_{I-1}, \theta_I) = k_{12}$

Inasmuch as the conditions 3.3 and 3.4 at the center of the plate are given in terms of the functions rather than their derivatives, the corresponding finite difference equations for the base-plate differential equations are written for $j = 1$. The equations may be denoted

4.33 $\qquad H_{13}(\Phi_0, \Phi_1, \Phi_2) = k_{13}$

4.34 $\qquad H_{14}(\psi_0, \psi_1, \psi_2) = k_{14}$

The boundary conditions 3.3 and 3.4 in difference form are

4.35 $\qquad \Phi_0 = 0$

4.36 $\qquad \psi_0 = 1$

Substituting 4.35 and 4.36 in 4.33 and 4.34 gives two equations which we will denote as

4.37 $\qquad H_{15}(\Phi_1, \Phi_2) = k_{15}$

4.38 $\qquad H_{16}(\psi_1, \psi_2) = k_{16}$

In 4.25, 4.26, 4.31, 4.32, 4.37, and 4.38 we have four equations in the four boundary variables Φ_1, ψ_1, θ_I, and S_I, and two equations in the two junction variables Φ_J, ψ_J.

We next obtain $2(I - 1) + 2(J - 2)$ equations at interior points by evaluating 4.1 and 4.2 at $i = 1, 2, - - - -, I-1$ and 4.10 and 4.11 at $j = 2, 3, - - - -, J-1$.

When evaluating 4.1 and 4.2 at $i = 1$ we bring in θ_0 and S_0. In order that all points lying on the junction satisfy the junction conditions, these terms must be eliminated through 4.24 and 4.23 as was done in the case of 4.21 and 4.22. Thus for $i = 1$ we can denote 4.1 and 4.2 as

4.39 $\qquad H_{17}(\theta_1, \theta_2, \phi_J, S_1) = k_{17}$

4.40 $\qquad H_{18}(S_1, S_2, \psi_J, \theta_1) = k_{18}$

The remaining $2(I - 2)$ equations for the cone may be denoted

4.41 $\qquad G_{1i}(\theta_{i-1}, \theta_i, \theta_{i+1}, S_i) = k_{1i} \quad i = 2,3, - - - -, I-2$

4.42 $\qquad G_{2i}(S_{i-1}, S_i, S_{i+1}, \theta_i) = k_{2i} \quad i = 2,3, - - - -, I-2$

At the interior points of the base plate the $2(J - 2)$ equations 4.10 and 4.11 may be denoted

4.43 $\qquad G_{3j}(\phi_{j-1}, \phi_j, \phi_{j+1}) = k_{3j} \quad j = 2,3, - - - -, J-1$

4.44 $\qquad G_{4j}(\psi_{j-1}, \psi_j, \psi_{j+1}) = k_{4j} \quad j = 2,3, - - - -, J-1$

We now investigate the difference approximations for the joined cones at their boundaries and junction. At the junction, cone 1 has its Ith net point and cone 2 has its 0th net point. Substituting $i = 0$ and $i = I$ in 4.1 and 4.2 gives four relations which we will denote

4.45 $\qquad K_1(\theta_{-1}, \theta_0, \theta_1, S_0) = h_1$

4.46 $\qquad K_2(S_{-1}, S_0, S_1, \theta_0) = h_2$

$$4.47 \qquad K_3(\theta_{I-1}, \ \theta_I, \ \theta_{I+1}, \ S_I) = h_3$$

$$4.48 \qquad K_4(S_{I-1}, \ S_I, \ S_{I+1}, \ \theta_I) = h_4$$

In these four equations we have θ_{-1}, S_{-1}, θ_{I+1}, and S_{I+1} which lie outside the junction. These four variables can be eliminated by means of junction conditions 3.19 and 3.21 which in difference form are

$$4.49 \qquad \left[h_I \ S_I \left(2\frac{h_{I+1} - h_{I-1}}{2\Delta s} - \frac{\nu \ h_I}{S_I} + h_I^2 \left(\frac{S_{I+1} - S_{I-1}}{2\Delta s}\right) \right]_1$$

$$= \left[h_o \ S_o \left(2\frac{h_1 - h_{-1}}{2\Delta s} - \frac{\nu h_o}{S_o}\right) + h_o^2 \left(\frac{S_1 - S_{-1}}{2\Delta s}\right) \right]_2$$

$$4.50 \qquad \left[\frac{E \ h_I^3}{12(1 - \nu^2)} \left(\frac{\theta_{I+1} - \theta_{I-1}}{2\Delta s} + \frac{\nu}{S_I} \ \theta_I\right) \right]_1$$

$$= \left[\frac{E \ h_o^3}{12(1 - \nu^2)} \frac{\theta_1 - \theta_{-1}}{2\Delta s} + \frac{\nu}{S_o} \ \theta_o\right) \right]_2$$

Solving 4.46 and 4.48 for S_{-1} and S_{I+1} respectively and substituting in 4.49 gives one equation in the variables S_o, S_1, θ_o, S_{I-1}, S_I, and θ_I. Solving 4.45 and 4.47 for θ_{-1} and θ_{I+1} respectively and substituting in 4.50 gives one equation in the variables θ_o, θ_1, S_o, θ_{I-1}, θ_I, and S_I. These

two equations may be denoted

4.51 $\qquad K_5(S_o,\ S_1,\ \theta_o,\ S_{I-1},\ S_I,\ \theta_I) = h_5$

4.52 $\qquad K_6(\theta_o,\ \theta_1,\ S_o,\ \theta_{I-1},\ \theta_I,\ S_I) = h_6$

We next replace θ_o by θ_I through junction condition 3.20 and we replace S_o by S_I through junction condition 3.22. Equations 4.51 and 4.52 then become

4.53 $\qquad K_7(S_1,\ S_{I-1},\ S_I,\ \theta_I) = h_7$

4.54 $\qquad K_8(\theta_1,\ \theta_{I-1},\ \theta_I,\ S_I) = h_8$

At the free end of cone 1 we may denote equations 4.1 and 4.2 in finite difference form as

4.55 $\qquad K_9(\theta_{-1},\ \theta_o,\ \theta_1,\ S_o) = h_9$

4.56 $\qquad K_{10}(S_{-1},\ S_o,\ S_1,\ \theta_o) = h_{10}$

Boundary conditions 3.15 and 3.17 in finite difference form are

4.57 $\qquad \dfrac{E\,h_o{}^3}{12(1 - \nu^2)}\left(\dfrac{\theta_1 - \theta_{-1}}{2\Delta s} + \dfrac{\nu}{S_o}\,\theta_o\right) = M_1$

4.58 $\qquad h_o\left(2\dfrac{h_1 - h_{-1}}{2\Delta s} - \dfrac{\nu h_o}{S_o}\right) S_o + h_o{}^2\left(\dfrac{S_1 - S_{-1}}{2\Delta s}\right) = \epsilon_1$

Solving 4.57 and 4.58 for θ_{-1} and S_{-1} respectively and substituting in 4.55 and 4.56 gives 2 equations which we will denote

4.59 $\qquad K_{11}(\theta_0, \theta_1, S_0) = h_{11}$

4.60 $\qquad K_{12}(S_0, S_1, \theta_0) = h_{12}$

At the free end of cone 2 we have two equations (see 4.31 and 4.32) which we may denote

4.61 $\qquad K_{13}(\theta_I, \theta_{I-1}, S_I) = h_{13}$

4.62 $\qquad K_{14}(S_I, S_{I-1}, \theta_I) = h_{14}$

In 4.53, 4.54, 4.59, 4.60, 4.61, and 4.62 we have four equations in the four boundary variables θ_0 and S_0 of cone 1 and θ_I and S_I of cone 2 and we have two equations in the two junction variables θ_I and S_I of cone 1. We next obtain $2(I - 1)$ equations at the interior points of cone 1 and $2(1 - 2)$ equations at the interior points of cone 2. When evaluating 4.1 and 4.2 at $i = 1$ we bring in, as mentioned previously, θ_0 and S_0. In the case of cone 1, however, we already have equations in θ_0 and S_0 (i.e., 4.59 and 4.60) and therefore the equations at interior points may be denoted

4.63 $F_{11}(\theta_{1-1}, \theta_1, \theta_{1+1}, S_1) = h_{11}$ $1 = 1, 2, - -, I-1$

4.64 $F_{21}(S_{1-1}, S_1, S_{1+1}, \theta_1) = h_{21}$ $1 = 1, 2, - -, I-1$

In the case of cone 2 we eliminate θ_o and S_o through junction conditions 3.20 and 3.22 to give

4.65 $F_{31}(\theta_I, \theta_1, \theta_2, S_1) = h_{31}$

4.66 $F_{41}(\theta_{1-1}, \theta_1, \theta_{1+1}, S_1) = h_{41}$ $1 = 2, 3, - -, I-1$

4.67 $F_{51}(S_I, S_1, S_2, \theta_1) = h_{51}$

4.68 $F_{61}(S_{1-1}, S_1, S_{1+1}, \theta_1) = h_{61}$ $1 = 2, 3, - -, I-1$

V. METHOD OF SOLUTION OF THE DIFFERENCE EQUATIONS

We can see the form of the difference equations for the cone and base-plate equations if we define the following matrices

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ - \\ - \\ - \\ - \\ - \\ \theta_I \end{bmatrix} \qquad S = \begin{bmatrix} S_1 \\ S_2 \\ - \\ - \\ - \\ - \\ S_I \end{bmatrix}$$

$$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ - \\ - \\ - \\ - \\ \Phi_J \end{bmatrix} \qquad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ - \\ - \\ - \\ - \\ \psi_J \end{bmatrix}$$

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The equations of the previous section can thus be written in
the form

$$5.1 \qquad A \begin{bmatrix} \theta \\ S \\ \Phi \\ \psi \end{bmatrix} = R$$

The matrices A and R can be formed from equations 4.25, 4.26,
4.31, 4.32, 4.37, 4.38, 4.39, 4.40, 4.41, 4.42, 4.43, and
4.44. The matrix A is symmetric and its diagonal matrices
are tri-diagonal; the disposition of its elements is shown
on page **27**. The equations corresponding to the elements of
each row are given to the left of the matrix. An asterisk
indicates non-zero constant coefficients and η indicates the
presence of the non-linear term due to the coupling introduced
by 2.15. The matrices A_{11}, A_{12}, A_{21}, A_{22} are square with I
rows and columns; the matrices A_{33} and A_{44} are square with
J rows and columns. The remaining matrices are rectangular
with I by J and J by I rows and columns.

The solution of the set of equations 5.1 could be ob-
tained easily if it were not for the non-linearity. Hence
we consider an iterative solution which assumes that $\sigma_r(a)$
in 2.15 is known. If we let the superscript n designate the
nth iteration, we proceed as follows: 1) choose the initial
$\sigma_r(a)$ as $(\sigma_r)^0$, 2) solve the linear system and calculate a
new value for $\sigma_r(a)$ from 2.15 using $\psi(a)$, 3) use the new value
of $\sigma_r(a)$ to calculate a new value of $\psi(a)$, 4) repeat the
above process until desired convergence is obtained.

We would expect this iterative scheme to converge because $\sigma_r(a)\Phi_j$ is relatively small. If the scheme does not converge, some linear extrapolation or interpotation between successive iterations may produce convergence. We leave this question to a future investigation.

Finally we need only give a practical method of solving the linear equations that arise in the iterative scheme. Since the diagonal matrices are tri-diagonal, we can use the method of factoring the matrix into a lower and upper matrix. This factorization requires the inversion of the A_{11}, A_{22}, A_{33}, and A_{44}. The resulting equations can be easily solved using a digital computer [3].

In the case of the joined cones we define the following matrices

$$\theta_1 = \begin{bmatrix} (\theta_1)_0 \\ (\theta_1)_1 \\ - \\ - \\ - \\ (\theta_1)_{I_1} \end{bmatrix} \qquad S_1 = \begin{bmatrix} (S_1)_0 \\ (S_1)_1 \\ - \\ - \\ - \\ (S_1)_{I_1} \end{bmatrix}$$

$$\theta_2 = \begin{bmatrix} (\theta_2)_1 \\ (\theta_2)_2 \\ - \\ - \\ - \\ (\theta_2)_{I_2} \end{bmatrix} \qquad S_2 = \begin{bmatrix} (S_2)_1 \\ (S_2)_2 \\ - \\ - \\ - \\ (S_2)_{I_2} \end{bmatrix}$$
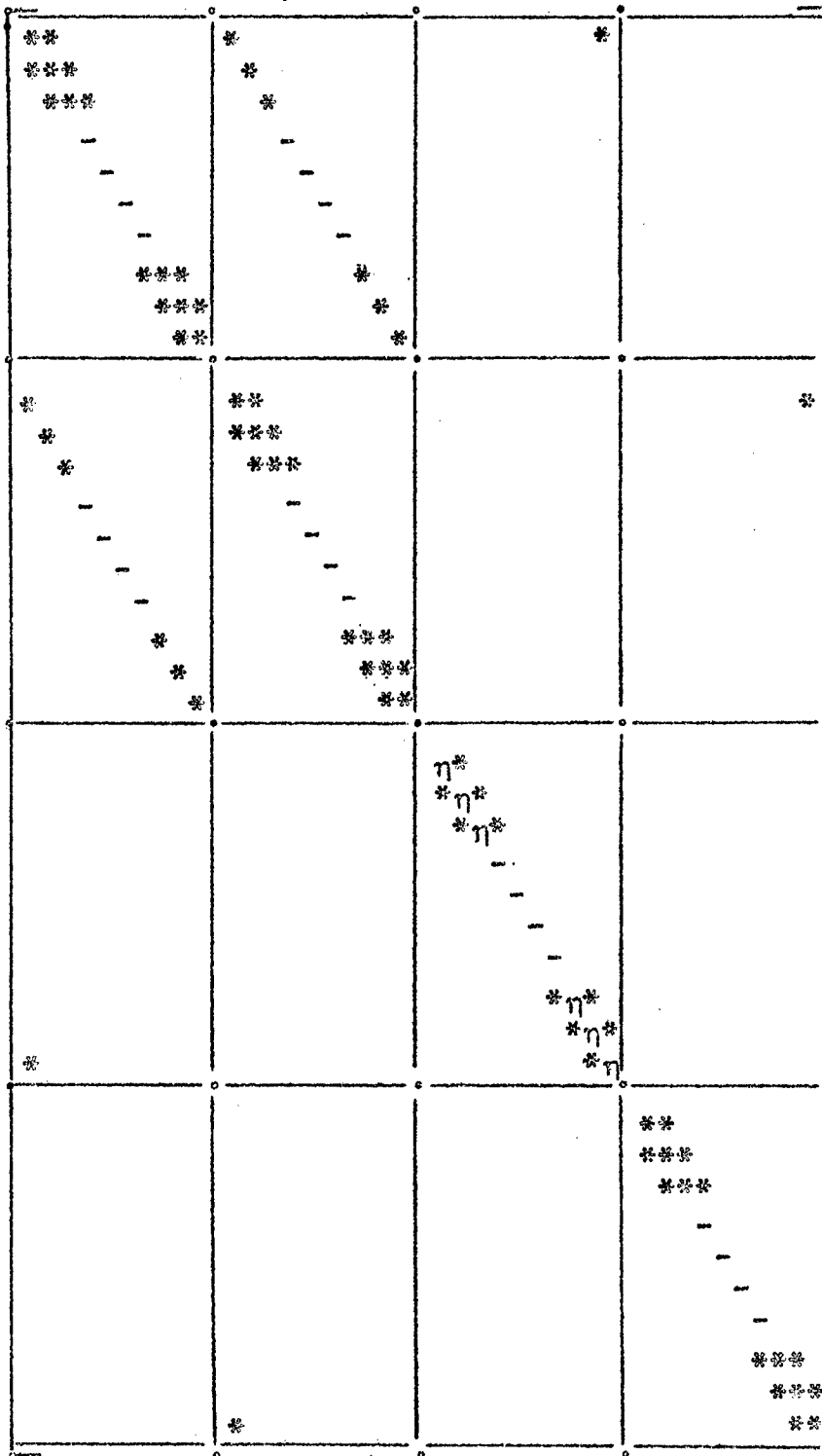
The appropriate equations can then be written

$$B \begin{bmatrix} \theta_1 \\ S_1 \\ \theta_2 \\ S_2 \end{bmatrix} = T$$

The matrices B and T can be formed from equations 4.53, 4.54, 5.49, 4.60, 4.61, 4.62, 4.63, 4.64, 4.65, 4.66, 4.67, and 4.68. The matrix B is symmetric and its diagonal matrices are tri-diagonal; the disposition of elements is shown on page 28. The equations corresponding to the elements of each row are given to the left of the matrix. An asterisk indicates non-zero constant coefficients. Since all equations are linear their solution can be carried out by a single application of the method described previously for an iterative cycle.
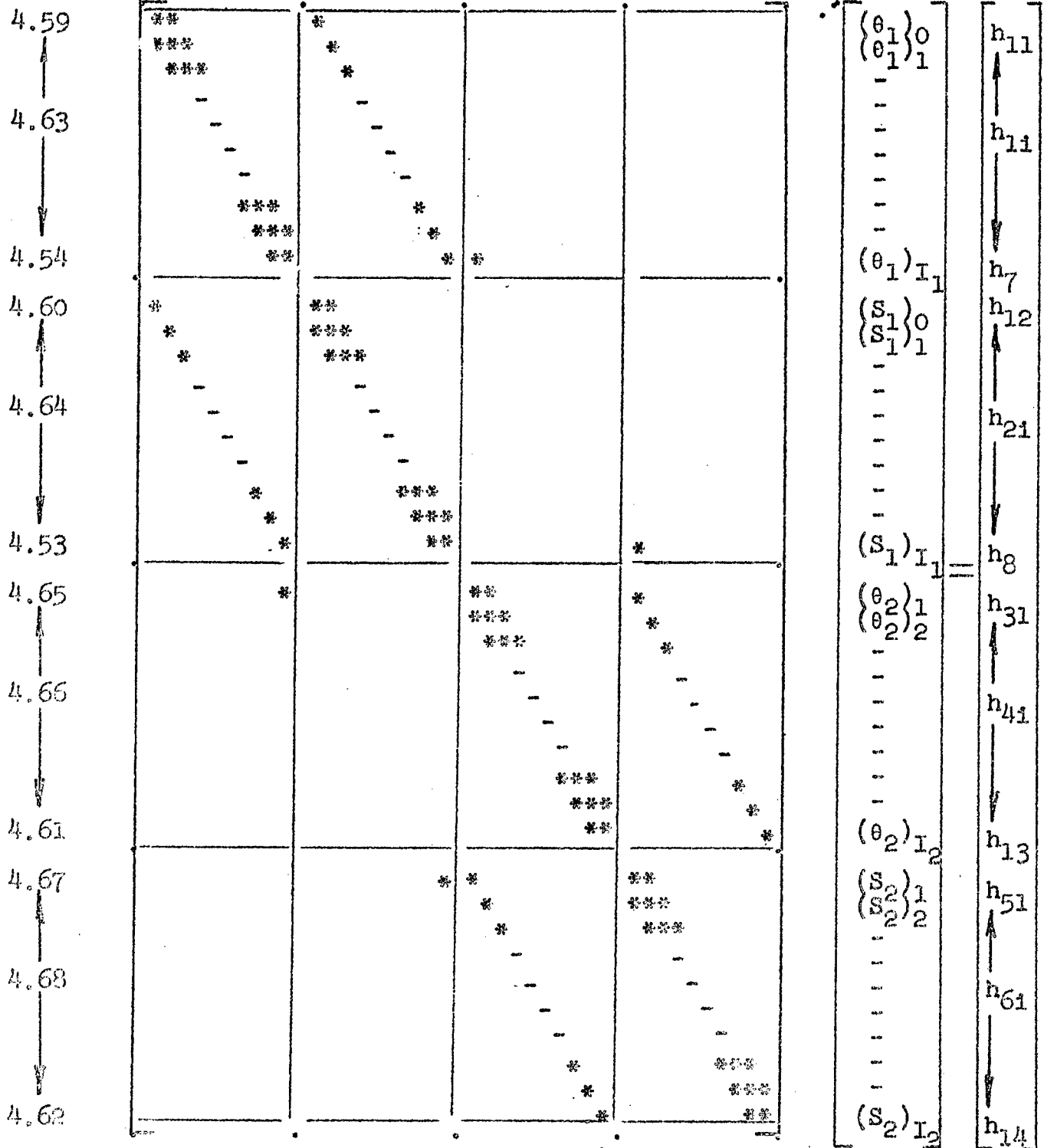
$$
\begin{bmatrix}
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
\end{bmatrix}
\begin{bmatrix}
\{ \theta_1 \}_1^0 \\
\vdots \\
(\theta_1)_{I_1} \\
\{ S_1 \}_1^0 \\
\vdots \\
(S_1)_{I_1} \\
\{ \theta_2 \}_2^1 \\
\vdots \\
(\theta_2)_{I_2} \\
\{ S_2 \}_2^1 \\
\vdots \\
(S_2)_{I_2}
\end{bmatrix}
=
\begin{bmatrix}
h_{11} \\
\updownarrow \\
h_{11} \\
h_7 \\
h_{12} \\
h_{21} \\
h_8 \\
h_{31} \\
h_{41} \\
h_{13} \\
h_{51} \\
h_{61} \\
h_{14}
\end{bmatrix}
$$

Row labels (left):
4.59, 4.63, 4.54, 4.60, 4.64, 4.53, 4.65, 4.66, 4.61, 4.67, 4.68, 4.62

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